Linear Algebra, Winter 2022

List 8

Polynomials

A **polynomial in x** is a function that can be written in the form

$$\_\_x^n + \_\_x^{n-1} + \cdots + \_\_x^2 + \_\_x + \_\_,$$

where each blank—called a **coefficient**—is a real or complex number, possibly including zero. A **real polynomial** is one whose coefficients are real numbers. In general, variables other than x can also be used (when complex numbers are involved, it is common, but *not* required, to use the variable z).

The **degree** of $f(x)$ is the highest power of x that has a non-zero coefficient.

187. Which of the following are polynomials (in any variable)?

(a) $8x^2 + 4x + 1$
(b) $8z^2 + 4z + 1$
(c) $x^{10} + 5x^6 - 100x$
(d) $(z^5 - 2z + 1)(z + 1)$
(e) $(z^5 - 2z + 1)\sin(z)$
(f) $3x^2 + 3x^{1/2} - 4$
(g) $x^2 + 2^x$
(h) $\sqrt{x^4 + 2x^2 + 1}$
(i) $z + \overline{z}$

188. Which of the following are real polynomials (in any variable)?

(a) $8x^2 + 4x + 1$
(b) $8z^2 + 4z + 1$
(c) $z^2 + 1$
(d) $z^2 + i$
(e) $(2 + i)x + (4 - i)$
(f) $(z + i)(z - i)$

189. For each of the following, give the degree if the expression is a polynomial in x, and otherwise write "not a polynomial".

(a) $\frac{5}{2}x^3 - 7x + 8$
(b) $9x^{10}$
(c) $6x^5 + \frac{1}{3}x + 5x^{-2}$
(d) $3x^2 + \sin(x)$
(e) $(x^2 + 2x - 1)^3$
(f) $5x$
(g) 5
☆(h) 0
(i) $\dfrac{8x + 1}{2x}$
(j) $\dfrac{x^3 + 7x}{2}$

The number c is a **zero** (also called a **root**) of the polynomial $f(x)$ if $f(c) = 0$.

190. Find all the zeroes of $2x^2 + x - 15$.

191. A cannonball fired at 400 m/s at an angle of 52° will have an initial vertical velocity of $400\sin(52°) \approx 315.2$ m/s, and it will have a height of

$$h(t) = \tfrac{-9.8}{2}t^2 + 315.2\,t$$

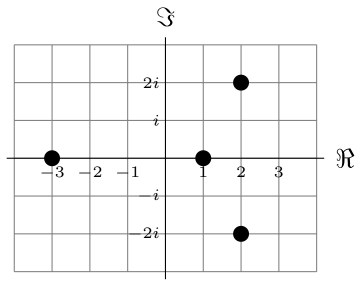
meters after t seconds. How many seconds will it take for the cannonball to reach the ground?

192. Find all the roots of $x^5 - 6x^4 + 34x^3$.

193. Given that 4 is one zero of $z^3 - 4z^2 + 49z - 196$, find all its roots.

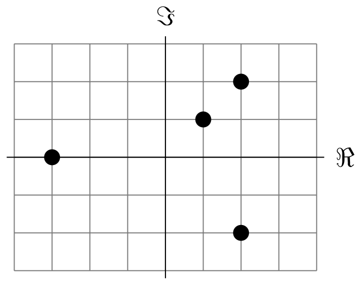
194. Given that $1 + 2i$ is a zero of $z^4 - 4z^3 + 12z^2 - 16z + 15$, find all its zeros.

195. The figure below shows four points on the complex plane.



Give an example of a polynomial of degree 4 whose roots are exactly these four points.

196. The figure below shows four points on the complex plane.



(a) Does there exist a polynomial with real coefficients whose roots are exactly these four points? Why or why not?

(b) Does there exist a polynomial with complex coefficients whose roots are exactly these four points? Why or why not?

197. Does there exist a polynomial $f(x)$ with *integer* coefficients for which...

(a) $f(35) = 0$ and $f(7) = 0$?

(b) $f(0) = 35$ and $f(7) = 0$?

(c) $f(0) = 53$ and $f(7) = 0$?

(d) $f(3i) = 5$ and $f(-3i) = 0$?

198. Does there exist a polynomial $f(x)$ with *real* coefficients for which...
(a) $f(35) = 0$ and $f(7) = 0$? (b)-(d) each other condition from Task 197?

199. Does there exist a polynomial $f(x)$ with *complex* coefficients for which...
(a) $f(35) = 0$ and $f(7) = 0$? (b)-(d) each other condition from Task 197?

200. Give the polynomial

$$f(x) = x^3 + \_\_x^2 + \_\_x + \_\_$$

for which $f(-1) = 0$, $f(3) = 0$, and $f(4) = 0$.

A polynomial is **reducible** if it can be factored into two non-constant polynomials; if not, it is **irreducible**. Whether a polynomial is reducible can depend on what kinds of numbers (e.g., real or complex) are allowed for the coefficients.

201. Are there polynomials $f(x)$ and $g(x)$ such that $f \cdot g = x^2 - 25$?
Is $x^2 - 25$ reducible or irreducible?

202. (a) Are there real polynomials f and g such that $f \cdot g = x^2 + 9$?

(b) Are there non-constant real polynomials f and g such that $f \cdot g = x^2 + 9$?

(c) Are there non-constant polynomials f and g such that $f \cdot g = x^2 + 9$?
Hint: $(3i)^2 = -9$.

☆203. (a) Is 19 prime? Why or why not?

(b) Is 13 prime? Why or why not?

(c) Is 11 prime? Why or why not?

(d) Re-read Task 177 and then re-read 203(b).

204. For the polynomial

$$f(z) = z^4 + z^3 + z^2 + 3z - 6 = (z^2 + z - 2)(z^2 + 3),$$

(a) find all roots of $f(z)$.

(b) find all zeros of $f(z)$.

(c) solve $f(z) = 0$ for complex z.

(d) factor $f(z)$ into linear complex factors.

(e) factor $f(z)$ into irreducible complex factors.

205. Factor the following polynomials into irreducible real factors:

(a) x^3+x^2+x+1 (b) x^3+x^2-x-1 (c) x^4-4x^3+8x (d) x^4+5x^2+6

206. Factor the following polynomials into irreducible complex factors:

(a) z^3+z^2+z+1 (b) z^3+z^2-z-1 (c) z^4-4z^3+8z (d) z^4+5z^2+6

207. Factor the polynomials

$$F(x) = x^3 - 6x^2 - 27x + 140$$
$$P(x) = x^3 - 14x^2 + 74x - 136$$

into irreducible real polynomials, knowing that $F(4) = P(4) = 0$.

208. Factor the polynomials from Task 207 into irreducible complex polynomials.

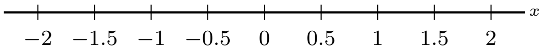
If r is root of the polynomial f, the **multiplicity** of r is the highest power m for which $(x - r)^m$ is a factor of f.

209. For $f(x) = (z - 3)^4(z + 2)$, what is the multiplicity of 3?

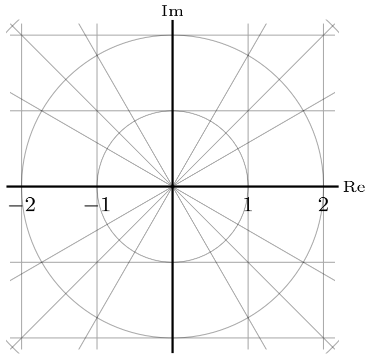
210. For $g(z) = z^3 + 2z^2 - 7z + 4$, what is the multiplicity of 1?

211. The only roots of $z^5 - 4z^4 + z^3 + 10z^2 - 4z - 8$ are -1 (with some multiplicity) and $+2$ (with some multiplicity). What is the sum of these multiplicities?

212. (a) On a real number line (like the blank one shown below), put a dot at every point x for which $x^6 = 1$.



(b) On a complex plane (like the blank one shown below), put a dot at every point z for which $z^6 = 1$.



213. (a) Find all the roots of $f(z) = 1 + z + z^2$.

(b) Find all the roots of $g(x) = 1 + x^2 + x^4$. (x can be complex)

(c) Find all the roots of $h(z) = 1 + z + z^2 + z^3 + z^4 + z^5$. Hint: $h(z) = \dfrac{1 - z^6}{1 - z}$.

☆214. Find the *sum* of all the roots (that is, add them together) of the polynomial

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9.$$